hep-th/0101194

Choptuik Scaling and Quasinormal Modes in the AdS/CFT Correspondence

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Abstract

We establish an exact connection between the Choptuik scaling parameter for the three-dimensional BTZ black hole, and the imaginary part of the quasinormal frequencies for scalar perturbations. Via the AdS/CFT correspondence, this leads to an interpretation of Choptuik scaling in terms of the timescale for return to equilibrium of the dual conformal field theory.

January 2001

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Within the context of numerical relativity, one of the most significant recent results is the evidence for universal scaling behaviour in black hole formation [1, 2]. In particular, one considers a generic smooth one-parameter family of initial data (labelled by p), such that a black hole is formed for values of p greater than a critical value p^*, while no black hole is formed for $p < p^*$. The mass of the black hole then satisfies the scaling relation [1]

$$M \sim (p - p^*)^\gamma, \tag{1}$$

where γ is a universal exponent known as the Choptuik scaling parameter. In [3], it was shown that the Gott time machine [4], namely a two-body collision process, gives a precise algebraic mechanism for the formation of the $(2+1)$-dimensional BTZ black hole. This led to an exact analytic determination of Choptuik scaling.

In [5]-[13], the quasinormal modes of scalar fields in the background of anti-de Sitter black holes were studied. The associated complex quasinormal frequencies describe the decay of the scalar perturbation, and depend only on the parameters of the black hole. In terms of the AdS/CFT correspondence [14]-[18], an off-equilibrium configuration in the bulk AdS space is related to an off-equilibrium state in the boundary conformal field theory. The timescale for the decay of the scalar perturbation is given by the imaginary part of the quasinormal frequencies. Thus, by virtue of the AdS/CFT correspondence, one obtains a prediction of the timescale for return to equilibrium of the dual conformal field theory. Interestingly, it was shown numerically [7] that the imaginary part of the quasinormal frequencies for intermediate-sized black holes, ω_{Im}, scaled with the horizon radius, r_+. In particular, it was found that

$$\omega_{\mathrm{Im}} \sim \frac{1}{\gamma} r_+, \tag{2}$$

where γ is the Choptuik scaling parameter. This relation, although not understood, suggested a deeper connection between black hole critical phenomena and quasinormal modes.

In this paper, we compute exactly the quasinormal modes of massive scalar fields in the background of the BTZ black hole, see also [11, 12]. It is shown that the imaginary part of the quasinormal frequencies has a universal scaling behaviour precisely of the form (2). This leads to a conformal field theory interpretation of Choptuik scaling within the context of the AdS/CFT correspondence.

To begin, we recall that the line element for the BTZ black hole can be written in the form [19, 20]

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2 . \tag{3}$$

The mass and angular momentum of the black hole can be expressed in terms of the inner and outer horizon radii, $r_\pm$, as

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+ r_-}{l}, \tag{4}$$

and we choose units for Newton's constant such that $8G = 1$.

We wish to study the properties of a massive scalar field in the background geometry of the BTZ black hole. A special feature of this $(2+1)$-dimensional case is that the corresponding

wave equation can be solved exactly in terms of hypergeometric functions [21, 22]. By choosing appropriate boundary conditions, we are led to an exact determination of the quasinormal modes for the scalar field. The scalar wave equation takes the form

$$\left(\nabla^2 - \frac{\mu}{l^2}\right)\Phi = 0, \tag{5}$$

where μ is the mass parameter. Using the ansatz,

$$\Phi = R(r)e^{-i\omega t}e^{im\phi}, \tag{6}$$

with the change of variables

$$z = \frac{r^2 - r_+^2}{r^2 - r_-^2}, \tag{7}$$

we are led to the radial equation

$$z(1-z)\frac{d^2R}{dz^2} + (1-z)\frac{dR}{dz} + \left(\frac{A}{z} + B + \frac{C}{1-z}\right)R = 0. \tag{8}$$

Here,

$$\begin{aligned}
A &= \frac{l^4}{4(r_+^2 - r_-^2)^2}(\omega r_+ - \frac{m}{l}r_-)^2, \\
B &= -\frac{l^4}{4(r_+^2 - r_-^2)^2}(\omega r_- - \frac{m}{l}r_+)^2, \\
C &= -\frac{\mu}{4}.
\end{aligned} \tag{9}$$

We now define

$$R(z) = z^\alpha(1-z)^\beta F(z). \tag{10}$$

The radial equation then assumes the standard hypergeometric form [23]

$$z(1-z)\frac{d^2F}{dz^2} + [c - (1+a+b)z]\frac{dF}{dz} - abF = 0, \tag{11}$$

where

$$\begin{aligned}
c &= 2\alpha + 1, \\
a + b &= 2\alpha + 2\beta, \\
ab &= (\alpha+\beta)^2 - B,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\alpha^2 &= -A, \\
\beta &= \frac{1}{2}(1 \pm \sqrt{1+\mu}).
\end{aligned} \tag{13}$$

Without loss of generality, we take $\alpha = -i\sqrt{A}$ and $\beta = \frac{1}{2}(1 - \sqrt{1+\mu})$.

In the neighbourhood of the horizon, $z = 0$, the two linearly independent solutions of (11) are given by [23] $F(a, b, c, z)$ and $z^{1-c}F(a - c + 1, b - c + 1, 2 - c, z)$. The quasinormal modes are defined as solutions which are purely ingoing at the horizon, and which vanish at infinity [7]. The solution which has ingoing flux at the horizon is given by

$$R(z) = z^{\alpha}(1 - z)^{\beta}F(a, b, c, z). \tag{14}$$

To implement the vanishing boundary condition at infinity, $z = 1$, we use the linear transformation formula [23]

$$\begin{aligned} R(z) &= z^{\alpha}(1-z)^{\beta}(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a, c-b, c-a-b+1, 1-z) \\ &+ z^{\alpha}(1-z)^{\beta}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, a+b-c+1, 1-z). \end{aligned} \tag{15}$$

Clearly, the first term in (15) vanishes. However, the vanishing of the second term imposes the restriction

$$c - a = -n, \quad \text{or} \quad c - b = -n, \tag{16}$$

where $(n = 0, 1, 2, ...)$. This condition leads directly to an exact determination of the quasinormal modes. From (12), we have

$$\begin{aligned} a &= \alpha + \beta + i\sqrt{-B}, \\ b &= \alpha + \beta - i\sqrt{-B}. \end{aligned} \tag{17}$$

Thus, we find that the left and right quasinormal modes, denoted by ω_L and ω_R, are given by

$$\begin{aligned} \omega_L &= \frac{m}{l} - 2i\left(\frac{r_+ - r_-}{l^2}\right)\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{1+\mu}\right), \\ \omega_R &= -\frac{m}{l} - 2i\left(\frac{r_+ + r_-}{l^2}\right)\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{1+\mu}\right). \end{aligned} \tag{18}$$

It is important to stress that this is an exact calculation of all quasinormal modes for the scalar field in a general BTZ background. The result (18) agrees with the special cases considered in [11, 12], for $\mu = 0$ and $J = 0$; quasinormal modes for the BTZ black hole were first studied in [5]. We also note that the imaginary parts of the quasinormal modes scale linearly with the left and right temperatures, defined by [24] $T_L = (r_+ - r_-)/2\pi l^2$ and $T_R = (r_+ + r_-)/2\pi l^2$.

The aim now is to determine the precise connection between these quasinormal modes and the Choptuik scaling parameter of the BTZ black hole. The first point to recall is that the BTZ black hole is defined as a quotient of AdS_3 by a discrete group of isometries of AdS_3 [25]. This is seen by noting [20] that AdS_3 can be viewed as the group manifold of $SL(2, \mathbf{R})$, with isometry group $(SL(2, \mathbf{R}) \times SL(2, \mathbf{R}))/Z_2$. Thus, for $\mathbf{X} \in SL(2, \mathbf{R})$, the isometry group acts by left and right multiplication, $\mathbf{X} \to \rho_L \mathbf{X} \rho_R$, with the identification $(\rho_L, \rho_R) \sim (-\rho_L, -\rho_R)$. The BTZ black hole is then defined as the quotient $\mathrm{AdS}_3/\langle(\rho_L, \rho_R)\rangle$, where the generators (ρ_L, ρ_R) are given by [20]

$$\rho_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/l} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/l} \end{pmatrix}, \quad \rho_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/l} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/l} \end{pmatrix}. \tag{19}$$

It is important to note that elements of $SL(2,\mathbf{R})$ are classified according to the value of their trace, namely [26]

$$\begin{aligned} |\,\mathrm{Tr}\,T\,| &< 2, \quad \text{Elliptic}, \\ |\,\mathrm{Tr}\,T\,| &= 2, \quad \text{Parabolic}, \\ |\,\mathrm{Tr}\,T\,| &> 2, \quad \text{Hyperbolic}. \end{aligned} \tag{20}$$

Thus, we see that the generators of the BTZ black hole are hyperbolic.

According to [27], a point particle spacetime in $(2+1)$ dimensions is defined via identifications by an elliptic generator. In [26, 28, 29], the formation of BTZ black holes from point particle collisions was investigated. In particular, it was shown [3] that the Gott time machine [4], suitably generalized to anti-de Sitter space, provides a precise mechanism for the formation of the BTZ black hole. Moreover, this purely algebraic process, in which a product of two elliptic generators becomes a hyperbolic generator, leads to an exact analytic determination of the Choptuik scaling parameter.

The Gott time machine is defined as a two-body collision process, with particles labelled by A and B, such that the mass and boost parameters obey a certain constraint, known as the Gott condition. The generator for each particle is defined in terms of its mass and boost parameters, denoted by α and ξ. Moreover, the effective generator for the two particles is given by the product [27, 30, 31], namely $T^G = T_B T_A$. The order parameter of interest is the trace of this generator, which takes the form [3]

$$\begin{aligned} \frac{1}{2}\,\mathrm{Tr}\,T^G &= -\cos\frac{\alpha_A}{2}\,\cos\frac{\alpha_B}{2} - \sin\frac{\alpha_A}{2}\,\sin\frac{\alpha_B}{2} \\ &+ \sin\frac{\alpha_A}{2}\,\sin\frac{\alpha_B}{2}\left[\cosh^2\left(\frac{\xi_A+\xi_B}{2}\right) + \cosh^2\left(\frac{\xi_A-\xi_B}{2}\right)\right] \\ &- \sin\frac{\alpha_A}{2}\,\sin\frac{\alpha_B}{2}\cos(\phi_A-\phi_B)\left[\cosh^2\left(\frac{\xi_A+\xi_B}{2}\right) - \cosh^2\left(\frac{\xi_A-\xi_B}{2}\right)\right]. \end{aligned} \tag{21}$$

The original Gott time machine is recovered by choosing particles with equal masses, and equal and opposite boosts, namely $\alpha_A = \alpha_B = \alpha, \xi_A = \xi_B = \xi, \phi_A - \phi_B = \pi$. Thus, when the Gott condition is satisfied, namely $\sin^2\frac{\alpha}{2}\,\cosh^2\xi > 1$, we see that T^G is a hyperbolic generator. When the Gott condition is not satisfied, we have an elliptic generator.

To construct the BTZ black hole, we simply take the independent left and right generators ρ_L, ρ_R to be defined in terms of two-particle Gott generators. Thus, we take $\rho_L = T^G$ in (21) with $\alpha_A = \alpha_B = \alpha, \phi_A - \phi_B = 0$. This gives

$$\frac{1}{2}\,\mathrm{Tr}\,\rho_L = \cosh\left(\frac{\pi}{l}(r_+ - r_-)\right) = -1 + 2\sin^2\frac{\alpha}{2}\,\cosh^2\left(\frac{\xi_A-\xi_B}{2}\right) \equiv p_L. \tag{22}$$

For the right generator, we choose $\rho_R = T^G$ with $\alpha_A = \alpha_B = \alpha, \phi_A - \phi_B = \pi$, leading to

$$\frac{1}{2}\,\mathrm{Tr}\,\rho_R = \cosh\left(\frac{\pi}{l}(r_+ + r_-)\right) = -1 + 2\sin^2\frac{\alpha}{2}\,\cosh^2\left(\frac{\xi_A+\xi_B}{2}\right) \equiv p_R. \tag{23}$$

We see that both ρ_L and ρ_R become hyperbolic if the input parameters α, ξ_A, ξ_B satisfy the appropriate Gott conditions, namely $p_L > 1$ and $p_R > 1$. Thus, the critical value of the input

parameters is $p_L^* = p_R^* = 1$. As shown in [3], the Choptuik scaling parameter can now be simply read off from (22) and (23), by using the formula, $\operatorname{arccosh} p = \ln[p + \sqrt{p^2 - 1}]$. Writing $p_L = p_L^* + \epsilon$, and $p_R = p_R^* + \epsilon$, we find to leading order

$$\begin{aligned} \frac{r_+ - r_-}{l} &= \frac{\sqrt{2}}{\pi}(p_L - p_L^*)^{1/2}, \\ \frac{r_+ + r_-}{l} &= \frac{\sqrt{2}}{\pi}(p_R - p_R^*)^{1/2}. \end{aligned} \tag{24}$$

Thus, the Choptuik scaling parameter for $(r_+ \pm r_-)$ is $\gamma = 1/2$. A scaling exponent of 1/2 was also found for collapsing dust shells in [32]. Other aspects of Choptuik scaling for the BTZ black hole have been investigated in [33]-[36].

We can now compare this result with the quasinormal frequencies (18). We see immediately that the negative of the imaginary part of ω_L and ω_R, denoted by $(\omega_L)_{\mathrm{Im}}$ and $(\omega_R)_{\mathrm{Im}}$, scales with $(r_+ - r_-)$ and $(r_+ + r_-)$, respectively. In particular, we have

$$\begin{aligned} (\omega_L)_{\mathrm{Im}} &= \frac{1}{\gamma}\left(\frac{r_+ - r_-}{l^2}\right)\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{\mu + 1}\right), \\ (\omega_R)_{\mathrm{Im}} &= \frac{1}{\gamma}\left(\frac{r_+ + r_-}{l^2}\right)\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{\mu + 1}\right). \end{aligned} \tag{25}$$

We have thus established an exact connection between the Choptuik scaling parameter and the imaginary part of the quasinormal modes. It is satisfying that in the $(2+1)$-dimensional case, these exact calculations lead to a result precisely of the form noticed in [7]. By virtue of the AdS/CFT correspondence, the imaginary part of the quasinormal modes has a direct interpretation in the dual conformal field theory. In the case at hand, the boundary conformal field theory of AdS_3 contains both left-moving and right-moving sectors [37], with Virasoro generators $\bar{L}_0 = (r_+ - r_-)^2/2l$ and $L_0 = (r_+ + r_-)^2/2l$, respectively. Thus, the return to equilibrium of the conformal field theory is specified in terms of the left and right timescales given by $\tau_L = 1/(\omega_L)_{\mathrm{Im}}$ and $\tau_R = 1/(\omega_R)_{\mathrm{Im}}$. Further analysis of BTZ black hole formation within the context of the AdS/CFT correspondence has been presented in [38]-[41].

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